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GREEDY MAXIMUM-CLIQUE DECOMPOSITIONS

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A greedy clique decomposition of a graph is obtained by removing maximal cliques from a graph one by one until the graph is empty. We have recently shown that any greedy clique decomposition of a graph of order n has at most $\frac{n^2}{4}$ cliques. A greedy max-clique decomposition is a particular kind of greedy clique decomposition where maximum cliques are removed, instead of just maximal ones. In this paper, we show that any greedy max-clique decomposition $\mathscr{C}$ of a graph of order n has $\sum\limits_{C\in\mathscr{C}} n(C)\le \frac{n^2}{2}$, where $n(C)$ is the number of vertices in C.

1. Introduction

For a graph G we denote its vertex set by $V(G)$, its edge set by $E(G)$ and their respective cardinalities by $n(G)$ and $m(G)$. By a *clique* of G we shall mean a complete subgraph of G, and by a *clique decomposition* of G we shall mean a collection of cliques which partition $E(G)$. An *ordered clique decomposition* of G is a pair $(\mathscr{C},\prec)$ where $\mathscr{C}$ is a clique decomposition of G and $\prec$ is a total ordering defined on $\mathscr{C}$. An ordered clique decomposition $(\mathscr{C},\prec)$ where $\mathscr{C}$ is attained by removing maximal cliques (i.e. their edges) one by one until the graph is empty, and $\prec$ coincides with the order in which cliques are removed, is called a *greedy clique decomposition*. A *greedy maximum-clique decomposition* is a particular kind of greedy clique decomposition where maximum cliques are chosen instead of just maximal ones.

A classic result of Erdős, Goodman, and Pósa [4] states any graph of order n has a clique decomposition with at most $\frac{n^2}{4}$ cliques. It was recently shown by the author [7] that any greedy clique decomposition of a graph of order n has at most $\frac{n^2}{4}$ cliques. This settled a conjecture of Winkler [9] who also conjectured [9]:

Conjecture 1.1. *For any greedy clique decomposition $(\mathscr{C},\prec)$ of a graph of order n, $\sum\limits_{C\in\mathscr{C}} n(C)\le \frac{n^2}{2}$.*

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It was shown by Chung [3] and independently by Győri and Kostochka [5] that for any graph of order n there exists a clique decomposition $\mathscr{C}$ such that

$$\sum_{C\in\mathscr{C}} n(C) \leq \frac{n^2}{2}.$$

Győri and Tuza [6] improved this result by showing that for $p\geq 4$ and any graph G of order n, there exists a clique decomposition $\mathscr{C}$ of G consisting solely of K_p's and edges where $\sum_{C\in\mathscr{C}} n(C)\leq 2t_{p-1}(n)$. Here $t_{p-1}(n)$ is the number of edges in the Turán graph on n vertices, having no K_p's.

In this paper, we prove Conjecture 1.1 in the special case for greedy max-clique decompositions. In the K_4-free case, this result verifies Conjecture 1.1 as greedy clique decompositions are the same as greedy max-clique decompositions in this case. The proof technique used here can also be used to verify Conjecture 1.1 for K_6-free graphs, but we have excluded a proof of this.

2. The main result

For a clique decomposition $\mathscr{C}$ of a graph G, and $i=2,3,\dots$ we let $\mathscr{C}^i$ denote the set of cliques of $\mathscr{C}$ of order i. For each vertex v let $\mathscr{C}_v$ denote the set of cliques of $\mathscr{C}$ containing v, and for each edge e we let $\mathscr{C}_e$ be the set of cliques of $\mathscr{C}$ containing an endpoint of e. For $i=2,3,4,\dots$ and for all $v\in V(G)$, and $e\in E(G)$, let $\mathscr{C}^i_v=\mathscr{C}^i\cap\mathscr{C}_v$ and $\mathscr{C}^i_e=\mathscr{C}^i\cap\mathscr{C}_e$.

Proposition 2.1. *(see [8]). For any greedy clique decomposition $(\mathscr{C},\prec)$ of a graph G and any edge $e\in\mathscr{C}^2$, it holds that $|\mathscr{C}_e|\leq n(G)-1$.*

Proof. Let $e=uv$, and define a set function $\psi:\mathscr{C}_e\to 2^{V(G)}$ where $\psi(e)=\{u,v\}$ and for all $C\in\mathscr{C}_e-\{e\}$, let $\psi(C)$ be the set of vertices of $V(C)-\{u,v\}$ which either belong to no other cliques of $\mathscr{C}_e-\{C\}$ or belong to some clique $D\in\mathscr{C}_e-\{C\}$ for which $D\prec C$. The sets $\psi(C)$, $C\in\mathscr{C}_e$ are seen to partition the set $U_{C\in\mathscr{C}_e}V(C)$, and thus

$$\sum_{C\in\mathscr{C}_e}(|\psi(C)|-1)+|\mathscr{C}_e| = |\sum_{C\in\mathscr{C}_e}V(C)| \leq n(G)$$

That is, $|\mathscr{C}_e|\leq n(G)-1-\sum_{C\in\mathscr{C}_e-\{e\}}(|\psi(C)|-1)$. Thus it suffices to show $\psi(C)\neq\emptyset$ for all $C\in\mathscr{C}_e-\{e\}$.

Suppose $C\in\mathscr{C}_e-\{e\}$. If $V(C)-U_{D\in\mathscr{C}_e\{C\}}V(D)\neq\emptyset$, then by definition of ψ, $\psi(C)\neq\emptyset$. Therefore, suppose that $V(C)\subseteq U_{D\in\mathscr{C}_e-\{C\}}V(D)$. Then $V(C)\cup\{u,v\}$ induces a clique C' of order $n(C)+1$. Let F be the first clique chosen into $\mathscr{C}$ which covers some edges of C'. Since, when each clique of $\mathscr{C}$ was chosen it was maximal, F cannot be properly contained in C', and thus $F\neq C$ and $F\neq e$. It then follows that $F\prec C$ and F meets C' at exactly one edge uy or vy, depending on whether $C\in\mathscr{C}_v$ or $C\in\mathscr{C}_u$, respectively. By definition of ψ, we see that $y\in\psi(C)$ and thus $\psi(C)\neq\emptyset$. ∎

For a clique decomposition $\mathscr{C}$ of a graph G we define a subgraph $G_{\mathscr{C}}$ of G by letting $V(G_{\mathscr{C}})=V(G)$ and letting $uv\in E(G_{\mathscr{C}})$ if $|\mathscr{C}_u|+|\mathscr{C}_v|\le n(G)$. We call a vertex v *positive* with respect to $\mathscr{C}$ if $|\mathscr{C}_v|>\frac{n(G)}{2}$. The following was shown in [8]:

Proposition 2.2. *If there exists a matching in $G_{\mathscr{C}}$ which covers all positive vertices, then* $\sum_{C\in\mathscr{C}} n(C)\le\frac{n^2}{2}$.

Proof. Suppose we have a matching in $G_{\mathscr{C}}$ which covers vertices $W\subseteq V(G)$ and W contains all the positive vertices. By counting in two different ways we have

$$\begin{aligned}\sum_{C\in\mathscr{C}} n(C) &= \sum_{v\in V(G)} |\mathscr{C}_v| \\ &= \sum_{v\in W} |\mathscr{C}_v| + \sum_{v\in V(G)-W} |\mathscr{C}_v| \\ &\le \frac{|W|}{2}\cdot n + (n-|W|)\cdot\frac{n}{2} = \frac{n^2}{2}.\end{aligned}$$

The last line follows from the fact that for any edge $uv\in E(G_{\mathscr{C}})$ in the matching, $|\mathscr{C}_u|+|\mathscr{C}_v|\le n$. ∎

We now prove the main result:

Theorem 2.3. *For a greedy max-clique decomposition of a graph G of order n,* $\sum_{C\in\mathscr{C}} n(C)\le\frac{n^2}{2}$.

Proof. Let G be a graph of order n and let $\mathscr{C}$ be a greedy max-clique decomposition of G. We shall show that there exists a matching in $G_{\mathscr{C}}$ which covers all positive vertices of G. It will then follow by Proposition 2.2 that $\sum_{C\in\mathscr{C}} n(C)\le\frac{n^2}{2}$. Let S be a nonempty subset of positive vertices, and let $N(S)$ be the set of neighbours of S in $G_{\mathscr{C}}$ (i.e. lying outside S). We shall show that $|N(S)|\ge|S|+1$. Suppose the contrary is true; that is, $|N(S)|\le|S|$. Let $u\in S$. There are at most $\frac{n-|S|-|N(S)|}{2}$ cliques of $\mathscr{C}_u$ containing at least two vertices of $V(G)-S-N(S)$. Furthermore, there are at most $|N(S)|$ cliques of $\mathscr{C}_u$ containing vertices of $N(S)$. Thus there are at least $\frac{n}{2}+\frac{1}{2}-\left(\frac{n-|S|-|N(S)|}{2}+|N(S)|\right)\ge\frac{1}{2}$ (since $|S|\ge|N(S)|$) cliques of $\mathscr{C}_u$ containing no vertices of $N(S)$ and at most one vertex of $V(G)-S-N(S)$. Let K be such a clique. We shall show by induction on k that $K\notin\mathscr{C}_u^k$ for $k=2,3,4,\ldots$,

If $K\in\mathscr{C}_u^2$, and $K=uv$, then Proposition 2.1 implies $uv\in E(G_{\mathscr{C}})$. Now $v\notin S$, for otherwise $|\mathscr{C}_u|+|\mathscr{C}_v|\ge n+1$. Thus $v\in N(S)$, but K was assumed to have no vertices in $N(S)$, and thus we have a contradiction. Thus $K\notin\mathscr{C}_u^2$.

Let $k\ge3$, and assume that no clique $C\in\mathscr{C}$ exists for which $n(C)<k$, and for which all vertices of C lie in S except for possibility one which is in $V(G)-N(S)$. Label the vertices of K by $u=u_1,u_2,u_3,\ldots,u_k$, where $u_1,u_2,\ldots,u_{k-1}\in S$ but $u_k\in V(G)-N(S)$. Let $e_i=u_1u_{i+1}$, $i=1,2,\ldots,k-1$.

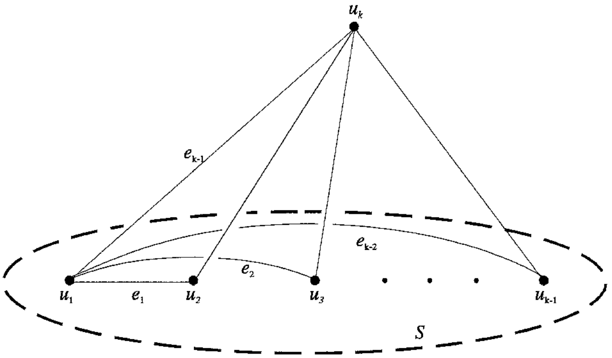

Fig. 2.1

Since $u_1, u_2, \ldots, u_{k-1}$ are all positive and u_k is either positive or belongs to $V(G) - S - N(S)$, we have that $e_i \notin E(G_{\mathscr{C}})$, $i = 1, \ldots, k-1$.

For convenience let $\mathscr{D}_i = \mathscr{C}_{u_i}$, $i = 1, \ldots, k$ and $\mathscr{D}_i^j = \mathscr{C}_{u_i}^j$, $j = 2, 3, \ldots$. Let $\mathscr{D} = \bigcup_i \mathscr{D}_i$. We have that

$$|\mathscr{C}_{e_i}| = |\mathscr{D}_1| + |\mathscr{D}_{i+1}| - 1 \geq n, \quad i = 1, \ldots, k-1$$

$$(k-1)n \leq \sum_{i=1}^{k-1} |\mathscr{C}_{e_i}| = (k-1)|\mathscr{D}_1| + \sum_{i=2}^{k-1} |\mathscr{D}_i| - (k-1). \tag{2.1}$$

We now define a set function $\Theta : \mathscr{D} \to 2^{V(G)}$ as follows: Let $\Theta(K) = V(K)$. For $C \in \mathscr{D} - \{K\}$ let $\Theta(C)$ be the set of vertices $v \in V(C) - V(K)$ for which there do not exists $k-1$ cliques $C_1, C_2, \ldots, C_{k-1} \in \mathscr{D} \cap \mathscr{C}_v$ for which $C \prec C_i$, $i = 1, \ldots, k-1$.

Lemma 2.4. *$|\Theta(C)| = n(C) - 1$ if $n(C) \leq k$, and $|\Theta(C)| \geq k - 1$ if $n(C) > k$.*

Proof. $|\Theta(K)| = n(K) = k$. So suppose $C \in \mathscr{D} - K$.

Case 1.

Suppose $n(C) \leq k$. Let $v \in V(C) - V(K)$ and suppose $C_1, C_2, \ldots, C_{k-1} \in \mathscr{D} - C$ contains v and $C \prec C_i$ for $i = 1, \ldots, k-1$. Then $v \cup V(K)$ induces a clique K' of G with $n(K') = k+1$, and $n(K') > n(K) \geq n(C) \geq n(C_i)$, $i = 1, \ldots, k-1$. Since $\mathscr{C}$ is a greedy max-clique decomposition, at least one clique of $\mathscr{C}$ which intersects K' has order at least $k+1$. But K, C and C_i, $i = 1, \ldots, k-1$ are the only cliques intersecting K', and all have order at most k; a contradiction. Thus $V(C) - V(K) \subseteq \Theta(C)$, and $|\Theta(C)| = n(C) - 1$.

Case 2.

Suppose $n(C) > k$. Suppose there exists $n(C) - k + 1$ vertices

$$v_1, v_2, \ldots, v_{n(C)-k+1} \in V(C) - V(K)$$

having the same property as v in Case 1. Then $\{v_1, v_2, \ldots, v_{n(C)-k+1}\} \cup V(K)$ induces a clique K' with $n(K') = k + n(C) - k + 1 = n(C) + 1$. As before, the only cliques of $\mathscr{C}$ intersecting K' will have order less than $n(K')$, and this contradicts the nature of $\mathscr{C}$. Thus $|\Theta(C)| \geq n(C) - 1 - (n(C) - k) = k - 1$. ∎

Let $X = V(G) - S - N(S)$, $Z = (\bigcup_{C \in \mathscr{D}_1} \Theta(C)) \cap N(S)$.

Let $x = |X|$, $z = |Z|$, $s = |S|$, and $s' = |N(S)|$.

For $L \subseteq V(G)$ let $1_L : V(G) \to \mathbb{R}$ be the indicator function

$$1_L(v) = \begin{cases} 1, & \text{if } v \in L \\ 0, & \text{otherwise} \end{cases}$$

For two functions $f : V(G) \to \mathbb{R}$ and $g : V(G) \to \mathbb{R}$ we shall write $f \leq g$ if $f(v) \leq g(v)$ for all $v \in V(G)$

Let $F : V(G) \to \mathbb{R}$ be defined by $F = (k-1)(1_S + 1_{N(S)}) + \binom{k}{2} 1_X$.

For each $C \in \mathscr{D}$ define $f_C : V(G) \to \mathbb{R}$ as follows:

Let $f_K = 1_{V(K)} = 1_{\Theta(K)}$.

For $C \in \mathscr{D}_k - \{K\}$ let $f_C = 1_{\Theta(C)} \cdot (1_S + 1_{N(S)} + (k-1) 1_X)$.

For $C \in \mathscr{D} - \mathscr{D}_k$ let $f_C = 1_{\Theta(C)} \cdot (1_S + 1_{N(S)} + \binom{k-1}{2} 1_X)$.

Proposition 2.5. *For all $v \in V(G) - V(K)$, $\sum_{C \in \mathscr{D}} f_C(v) \leq F(v)$ and for all $v \in V(K)$, $\sum_{C \in \mathscr{D}} f_C(v) = 1 \leq F(v) - (k-2)$.*

Proof. The second assertion is straightforward, so we shall prove the first. We have

$$\sum_{C \in \mathscr{D}} f_C \leq (1_S + 1_{N(S)}) \sum_{C \in \mathscr{D}} 1_{\Theta(C)}$$

$$+ 1_X \left((k-1) \sum_{C \in \mathscr{D}_k} 1_{\Theta(C)} + \binom{k-1}{2} \sum_{C \in \mathscr{D} - \mathscr{D}_k} 1_{\Theta(C)} \right).$$

By definition of Θ, $\sum_{C \in \mathscr{D}} 1_{\Theta(C)} \leq (k-1) 1_{V(G)}$. Thus

$$(1_S + 1_{N(S)}) \sum_{C \in \mathscr{D}} 1_{\Theta(C)} \leq (k-1)(1_S + 1_{N(S)}).$$

We also see that

$$1_X\left((k-1)\sum_{C\in\mathscr{D}_k}1_{\Theta(C)}+\binom{k-1}{2}\sum_{C\in\mathscr{D}-\mathscr{D}_k}1_{\Theta(C)}\right)$$
$$\le 1_X\cdot\left((k-1)+(k-2)\binom{k-1}{2}\right)\cdot 1_X=\binom{k}{2}1_X.$$

Comparing the above with F yields the proposition. ∎

Let $\mathscr{D}_{N(S)}=\{C\in\mathscr{D}:\Theta(C)\cap N(S)\neq\emptyset\}$.

Proposition 2.6. *$\sum_v f_C(v)\ge 1$ for all $C\in\mathscr{D}$, and furthermore, $\sum_v f_C(v)\ge k-1$ for all $C\in\mathscr{D}-\mathscr{D}_{N(S)}$.*

Proof. The first assertion holds since

$$\sum_v f_C(v)\ge\sum_v 1_{\Theta(C)}(v)\ge|\Theta(C)|,\text{ and}$$
$$|\Theta(C)|\ge\min\{n(C)-1,\,k-1\}\ge 1,\text{ by Lemma 2.1.}$$

The above, together with Lemma 2.4 also shows that if $n(C)\ge k$, then $\sum_v f_C(v)\ge|\Theta(C)|\ge k-1$. We can thus assume that $C\in\mathscr{D}-\mathscr{D}_{N(S)}$, $n(C)<k$, and hence also that $|\Theta(C)|=n(C)-1$ and $\Theta(C)=V(C)-V(K)$ (by Lemma 2.4).

Suppose $C\in\mathscr{D}_k$. Then $f_C=1_{\Theta(C)}(1_S+1_{N(S)}+(k-1)1_X)$. Since $n(C)<k$, and $V(C)\cap N(S)=\Theta(C)\cap N(S)=\emptyset$ (because $C\notin\mathscr{D}_{N(S)}$), it follows by the inductive assumption that $|V(C)\cap(V(G)-S)|=|V(C)\cap X|\ge 2$. Thus $|\Theta(C)\cap X|=|(V(C)-\{u_k\})\cap X|\ge 1$, and therefore $\sum_v f_C(v)\ge(k-1)\cdot\sum_v 1_{\Theta(C)}(v)1_X(v)\ge k-1$.

Suppose $C\notin\mathscr{D}_k$; that is $C\in\mathscr{D}-\mathscr{D}_k-\mathscr{D}_{N(S)}$. Again, the induction hypothesis implies

$$|\Theta(C)\cap X|=|V(C)\cap X|\ge 2.$$

But now

$$\sum_v f_C(v)=\sum_v 1_{\Theta(C)}(v)\left(1_S(v)+1_{N(S)}(v)+\binom{k-1}{2}1_X(v)\right)$$
$$\ge\binom{k-1}{2}\sum_v 1_{\Theta(C)}(v)\cdot 1_X(v)\ge k-1.$$

This completes the proof. ∎

By Proposition 2.5, it follows that

$$\sum_v\sum_{C\in\mathscr{D}}f_C(v)\le\sum_v F(v)-k(k-2).$$

By Proposition 2.6 and the above, we deduce that

$$(2.2)\qquad (k-1)|\mathcal{D}-\mathcal{D}_{N(S)}|+|\mathcal{D}_{N(S)}|\leq \sum_{v}\sum_{C\in\mathcal{D}} f_C(v)\leq \sum_{v} F(v)-k(k-2).$$

But

$$(2.3)\qquad (k-1)n=\sum_{v} F(v)-\binom{k-1}{2}x.$$

Now (2.1) combined with (2.2) and (2.3) yields:

$$\begin{aligned}
&(k-1)|\mathcal{D}_1|+\sum_{i=2}^{k}|\mathcal{D}_i|-(k-1)\geq n(k-1)\\
&=\sum_{v} F(v)-\binom{k-1}{2}x\\
&\geq (k-1)|\mathcal{D}-\mathcal{D}_{N(S)}|+|\mathcal{D}_{N(S)}|+k(k-2)-\binom{k-1}{2}x.
\end{aligned}$$

Or

$$(2.4)\quad (k-2)|\mathcal{D}-\mathcal{D}_1-\mathcal{D}_{N(S)}|-(k-2)|D_1\cap\mathcal{D}_{N(S)}|+k(k-2)-\binom{k-1}{2}x\leq 0.$$

Since $u_iu_j\notin E(G_{\mathscr{C}})$ for $2\leq i<j\leq k$, we have $|\mathcal{D}_i|+|\mathcal{D}_j|\geq n+1$ for $2\leq i<j\leq k$. Thus summing these inequalities over i and j yields

$$(k-2)\sum_{i=2}^{k}|\mathcal{D}_i|\geq\binom{k-1}{2}(n+1)$$

or,

$$\sum_{i=2}^{k}|\mathcal{D}_i|\geq\frac{(k-1)(n+1)}{2}.$$

Thus

$$(2.5)\qquad |\mathcal{D}-\mathcal{D}_1|=\sum_{i=2}^{k}(|\mathcal{D}_i|-1)\geq\frac{(k-1)(n+1)}{2}-(k-1).$$

We have that $|\mathcal{D}-\mathcal{D}_1-\mathcal{D}_{N(S)}|=|\mathcal{D}-\mathcal{D}_1|-|(\mathcal{D}-\mathcal{D}_1)\cap\mathcal{D}_{N(S)}|$.

By definition of $\mathcal{D}_{N(S)}$, at most $k-1$ cliques of $\mathcal{D}_{N(S)}$ meet a given vertex of $N(S)$. By definition of Z, every vertex of Z meets a clique of $\mathcal{D}_1\cap\mathcal{D}_{N(S)}$. Thus, $|(\mathcal{D}-\mathcal{D}_1)\cap\mathcal{D}_{N(S)}|\leq(k-2)z+(k-1)(s'-z)$. This combined with (2.5) gives

$$|\mathcal{D}-\mathcal{D}_1-\mathcal{D}_{N(S)}|\geq\frac{(k-1)(n+1)}{2}-(k-1)-(k-2)z-(k-1)(s'-z).$$

Noticing that $|\mathscr{D}_1 \cap \mathscr{D}_{N(S)}| \le z$, the above combined with (2.4) yields:

$$\binom{k-1}{2}(n+1) - (k-1)(k-2) - (k-2)^2 z - (k-1)(k-2)s' + (k-1)(k-2)z$$
$$-(k-2)z + k(k-2) - \binom{k-1}{2}x \le 0.$$

Since we assumed in the beginning that $s = |S| \ge |N(S)| = s'$, it follows that $x = n - s - s' \le n - 2s'$. Thus we have from the above:

$$\binom{k-1}{2}(n+1) + k - 2 - (k-1)(k-2)s' - \binom{k-1}{2}(n-2s') \le 0,$$
$$\binom{k-1}{2} + k - 2 \le 0.$$

Thus $k < 2$, and this contradicts the inductive assumption that $k \ge 3$. It now follows by induction that $n(K) \ne k$ for $k = 2, 3, 4, \ldots$, and therefore K cannot exist; a contradiction. It must therefore be that our original assumption that $|S| \ge |N(S)|$ is incorrect, and $|S| < |N(S)|$. Since S was an arbitrarily chosen nonempty subset of positive vertices, it now follows from Hall's Theorem [1, 2] that there exists a matching in $G_{\mathscr{C}}$ covering all positive vertices. It thus follows from Proposition 2.2 that $\sum_{C \in \mathscr{C}} n(C) \le \frac{n^2}{2}$. ∎

Remark. In the above proof we showed that $|N(S)| > |S|$ for any nonempty subset of positive vertices. Suppose now we have a max-clique decomposition $\mathscr{C}$ of G for which $\sum_{C \in \mathscr{C}} n(C) = \frac{n^2}{2}$ where $n = n(G)$. Suppose there exists a vertex v for which $|\mathscr{C}_v| < \frac{n}{2}$. We have from the vertices, and thus (by Hall's Theorem) there exists a matching in $G_{\mathscr{C}} - v$ which covers all positive vertices. Thus we see (as in Proposition 2.2) that $\sum_{u \in V(G) - v} |\mathscr{C}_u| \le \frac{(n-1)n}{2} = \frac{n^2}{2} - \frac{n}{2}$ and $\sum_{u \in V(G)} |\mathscr{C}_u| < \frac{n^2}{2}$. But this contradicts the assumption that $\sum_{C \in \mathscr{C}} n(C) = \frac{n^2}{2}$. Thus it must be the case that $|\mathscr{C}_v| \ge \frac{n}{2}$ for each vertex v, and in fact that $|\mathscr{C}_v| = \frac{n}{2}$ for all v.

We have thus shown that any graph having a max-clique decomposition $\mathscr{C}$ for which $\sum_{C \in \mathscr{C}} n(C) = \frac{n^2}{2}$ must satisfy $|\mathscr{C}_v| = \frac{n}{2}$ for all v. In [8] we completely determined all such graphs G, where G has no complete graphs of order four. As yet, we have not determined all such graphs for the general case.

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